We propose a technique for performing quantum state tomography of photonic polarization-encoded multiqubit states. Our method uses a single rotating wave plate, a polarizing beam splitter, and two photon-counting detectors per photon mode. As the wave plate rotates, the photon counters measure a pseudocountinuous signal which is then Fourier transformed. The density matrix of the state is reconstructed using the relationship between the Fourier coefficients of the signal and the Stokes’ parameters that represent the state. The experimental complexity, i.e., different wave plate rotation frequencies, scales linearly with the number of qubits.

In linear optics, where quantum information is encoded in the polarization of a single photon, different measurement settings are realized with a combination of linear optical elements such as wave plates, beam splitters, and polarizing beam splitters, followed by photon counting. QST was first accomplished in such systems by White et al. [7], where the measurement settings corresponded directly to the Stokes’ parameters used to characterize the polarization state of the classical electromagnetic field [8]. Later it was suggested that an over-complete symmetric six-measurement set [9] or an informationally complete symmetric four-measurement set [9–13] be used for improved performance. Other extensions, such as those considering optimal experimental design under realistic technical constraints [14,15], or modifications due to inaccessible information [16–22] or preferable measurement choices [9,11,12,23–30] have also been considered.

To date, implementations of QST of photonic polarization-encoded qubits have utilized either multiple wave plates and/or multiple beam splitters per qubit. We propose a technique that encoded qubits have utilized either multiple wave plates and/or wave plate rotating at frequency $\Omega_1$. The experimental complexity of this method scales linearly with the number of qubits in terms of the number of settings.

### I. QUANTUM STATE TOMOGRAPHY

Tomography is the process of constructing a representation of an object by imaging it in different sections. In quantum state tomography, we aim to construct a representation of a quantum state $\rho$ from different measurement outcomes. An $n$-qubit system is specified by $4^n - 1$ real parameters. We therefore require at least this many outcomes of linearly independent measurements to specify $\rho$.

The probability of obtaining measurement outcome $j$, given a measurement operator $\hat{M}_j$, is given by

$$p_j = \langle \hat{M}_j \rangle = \text{Tr}[\rho \hat{M}_j] = \frac{n_j}{N_j},$$

where $n_j$ is the number of counts and $N_j$ is a constant dependent on the detector efficiency and duration of data collection. In a polarization-encoded linear optical system, any projective measurement can be realized with a quarter-wave plate, a half-wave plate, and a polarizing beam splitter, as shown in Fig. 1(a). A popular choice corresponds to the three Pauli operators.

We can always write the density matrix of an $n$-qubit system in terms of Hermitian operators $\hat{\sigma}_i$,

$$\rho = \frac{1}{2^n} \sum_{i_1, \ldots, i_n=0}^3 S_{i_1, \ldots, i_n} \hat{\sigma}_{i_1} \otimes \cdots \otimes \hat{\sigma}_{i_n},$$

where $\hat{\sigma}_0 = |H\rangle\langle H| + |V\rangle\langle V|$ is the identity operator and $\hat{\sigma}_{1,2,3}$ are the Pauli operators: $\hat{\sigma}_1 = |H\rangle\langle V| + |V\rangle\langle H|$, $\hat{\sigma}_2 = i(|V\rangle\langle H| - |H\rangle\langle V|)$, and $\hat{\sigma}_3 = |H\rangle\langle H| - |V\rangle\langle V|$. The coefficients $S_{i_1, \ldots, i_n} = \text{Tr}[\rho (\hat{\sigma}_{i_1} \otimes \cdots \otimes \hat{\sigma}_{i_n})]$ completely characterize the state. $S_{i_1, \ldots, i_n}$ are normalized generalizations of the classical parameters introduced by Stokes in 1852 [8], and will hereafter be simply referred to as Stokes’ parameters.

Quantum state preparation is an essential ingredient in the realization of quantum technologies such as quantum computing [1], quantum cryptography [2], and other quantum information protocols [3]. A crucial aspect of reliable state preparation is the ability to accurately characterize the state of a quantum system. To this end, quantum state tomography (QST) allows the reconstruction of a state’s density matrix of a quantum system. To this end, quantum state tomography (QST) allows the reconstruction of a state’s density matrix.
Combining Eqs. (1) and (2), we find a linear relationship between Stokes’ parameters and the probability \( p_j \):

\[
p_j = \frac{1}{2^n} \sum_{i_1, ..., i_n=0}^{3} S_{i_1, ..., i_n} \text{Tr} [\hat{\delta}_{i_1} \otimes \cdots \otimes \hat{\delta}_{i_n} \hat{M}_j],
\]

where \( \hat{M}_j \) acts on the entire multiqubit system. By making \( 4^n - 1 \) linearly independent measurements, it is possible to solve for Stokes’ parameters and reconstruct the density matrix according to Eq. (2). This can be achieved through a variety of methods, including simple linear inversion, least-squares estimation or the popular maximum likelihood estimation method [33]. Alternatively, one can look to a growing number of exciting new techniques such as the forced purity routine [34], Baysean mean estimation [35], von Neumann entropy maximization [37], hedged maximum likelihood estimation [38], minimax estimation [39], and techniques that focus on reconstructing the state with reliable error bars [40] and confidence regions [41].

II. FOURIER TRANSFORM TOMOGRAPHY

In this section, we show how the quantum state of a multiqubit system can be represented by a single joint-probability signal and how the measurement of this signal enables the reconstruction of the quantum state.

In our proposal, identical copies of the state are prepared and subsequently pass through a series of optical elements. For a multiphoton state, each photon mode \( m \) is incident on a single wave plate rotating at frequency \( \Omega_m \) followed by a polarizing beam splitter (PBS). Photon counters at the output ports of the PBS continuously measure the intensity, which can be processed to recover Stokes’ parameters. A schematic of this setup is shown in Fig. 1(b) for a single qubit and Fig. 1(c) for multiple qubits. For multiple qubits, the signal measured is a “coincidence intensity” corresponding to the joint probability of detecting photons at each PBS.

The time-dependent single-qubit projection-valued measure (PVM) associated with the probability of detecting a photon in the horizontal or vertical output modes of each PBS is given by \( \{ \hat{M}_m^H(t), \hat{M}_m^V(t) \} \) where

\[
\hat{M}_m^a(t) = \hat{U}_m^a |a\rangle \langle a| \hat{U}_m^a,
\]

for \( a = H, V \), where \( m \) labels the qubit mode and

\[
\hat{U}_m(t) = \cos \left( \frac{\beta}{2} \right) \hat{\sigma}_0 - i \sin \left( \frac{\beta}{2} \right) \vec{\nu}_m(t) \cdot \vec{\sigma}
\]

is the unitary operator associated with a wave plate in mode \( m \). \( \hat{U}_m(t) \) rotates the operators \(|a\rangle \langle a|\) on the Bloch sphere by an angle \( \beta \), about the vector, \( \vec{\nu}_m(t) \), where \( \vec{k} \) and \( \vec{\ell} \) are unit vectors in Euclidian space (defined by the axes in Fig. 2) and \( \vec{\nu} \cdot \vec{\sigma} = \nu_1 \sigma_1 + \nu_2 \sigma_2 + \nu_3 \sigma_3 \). As the wave plate rotates about the beam axis at frequency \( \Omega_m \) in real space, \( \vec{\nu}_m(t) \) rotates about the \( y \) axis in Euclidian space at frequency \( \omega_m = 2\Omega_m \). We assume that the fast axis of the wave plate is aligned at 0 degrees to the horizontal as defined by the polarization of the photons. A phase factor can be included in \( \vec{\nu}_m(t) \) to account for different initial alignment of the wave plate. The resulting projector \( \hat{M}_m^H(t) \) traces out a figure-eight path on the Bloch sphere, as shown in Fig. 2. The retardance of the wave plate \( \beta \) determines the size of the figure eight.

To characterize an \( n \)-qubit state, one measures a joint probability of detecting a photon in the \( H \) mode of each PBS. This is given by

\[
p_{a}(t) = \frac{1}{2^n} \sum_{i_1, ..., i_n=0}^{3} S_{i_1, ..., i_n} \chi_{1,i_1} \cdots \chi_{n,i_n},
\]
where

\[
\chi_{m,i} := \text{Tr}[\hat{\sigma}_i \hat{M}_m^H(t)],
\]

and therefore

\[
\chi_{m,0} = 1,
\]

\[
\chi_{m,1} = s^2 \sin(2\omega_m t),
\]

\[
\chi_{m,2} = 2cs \sin(\omega_m t),
\]

\[
\chi_{m,3} = c^2 + s^2 \cos(2\omega_m t),
\]

where \(c = \cos(\beta/2)\) and \(s = \sin(\beta/2)\).

Note that the choice of analyzing the signal from mode \(H\) rather than mode \(V\) is arbitrary and typically both modes will need to be measured to ensure normalized probabilities.

Without loss of generality, we restrict \(0 < \omega_1 < \ldots < \omega_n\).

For two qubits, \(\omega_2 = r\omega_1\) where \(r > 1\). If \(r\) is an irrational number, the signal does not have a finite period. If \(r\) is a rational number, we can write \(r = p/q\), where \(p\) and \(q\) are integers. In this case, the period of the two-qubit signal is given by

\[
T(r) = \frac{2\pi q}{\omega_1 \gcd(p,q)},
\]

where \(\gcd(p,q)\) is the greatest common denominator of \(p\) and \(q\). For \(n > 2\), the period of the signal can be determined via recursion. A shorter period is favorable from an experimental perspective which, for a constant \(\omega_1\), occurs when \(r\) is an integer. The lowest integer that ensures sufficient Fourier coefficients to solve for Stokes’ parameters is \(r = 5\).

In practice \(p_n\) will not be a continuous function of time but rather a discretized approximation. The discretized signal will be divided into time bins, with \(N\) coincidence counts in each bin. The number of time bins per period, \(N\), must be at least the Nyquist rate, i.e. twice the highest frequency contained within the signal, to avoid aliasing.

\[
\text{FIG. 2. (Color online) Path traced out by } \hat{M}_m^H(t), \text{ defined in Eq. (4), for } \beta = \pi/4 \text{ (green, dotted); } \beta = \pi/2 \text{ (blue, dashed); and } \beta = 11\pi/15 \text{ (orange, solid).}
\]
where the Fourier coefficients are given by
\begin{align}
    a_0 &= S_0 + S_{3c^2}^2; \\
    b_1 &= S_{2cs}; \\
    a_2 &= S_2^2 + 2; \\
    b_2 &= S_{1s^2}^2,
\end{align}
where \( c = \cos(\beta/2) \) and \( s = \sin(\beta/2) \). Linear inversion of Eq. (14) gives the Stokes’ parameters in terms of the Fourier coefficients:
\begin{align}
    S_0 &= a_0 - \frac{2a_2c^2}{s^2}; \\
    S_1 &= \frac{2b_2}{s^2}; \\
    S_2 &= \frac{b_1}{cs}; \\
    S_3 &= \frac{2a_2}{s^2}.
\end{align}
Substitution into Eq. (2), gives the density matrix in terms of the Fourier coefficients:
\[
    \rho_1 = \begin{pmatrix}
        \frac{1}{2} (a_0 + 2a_2) & \frac{b_1}{s} & \frac{1}{2} (a_0 + 2a_2) - \frac{2a_2}{s} \\
        \frac{b_1}{s} & i \frac{b_1}{s} & \frac{b_1}{s} \\
        \frac{1}{2} (a_0 + 2a_2) - \frac{2a_2}{s} & \frac{b_1}{s} & \frac{1}{2} (a_0 + 2a_2) - \frac{2a_2}{s}
    \end{pmatrix}.
\]

As an example, consider a single-qubit state \(|\psi\rangle_1\) that has experienced depolarizing noise, characterized by the parameter \( d \), such that
\[
    \hat{\rho}_1 = d|\psi\rangle_1\langle\psi|_1 + (1-d)\hat{\sigma}_z|\psi\rangle_1\langle\psi|_1\hat{\sigma}_z.
\]
Specifically, let’s consider
\[
    |\psi\rangle_1 = \frac{1}{\sqrt{2}} (|H\rangle + e^{-i\pi/4}|V\rangle),
\]
and \( d = 0.1 \). A retardance of \( \beta = 11\pi/15 \) produces the signal shown in Fig. 4(a). Performing a fast Fourier transform (FFT) of the discretized signal yields the Fourier coefficients in Fig. 4(a). The coefficients \( a_f \) and \( b_f \) correspond to the real and imaginary parts of the list generated by the FFT, respectively. Inserting the coefficients, \( a_0 = 1, b_1 = 0.210, a_2 = 0, \) and \( b_2 = -0.236 \) into the density matrix in Eq. (16) gives
\[
    \rho_1 = \begin{pmatrix}
        0.5 & 0.283 - 0.283i \\
        -0.283 + 0.283i & 0.5
    \end{pmatrix},
\]
which corresponds to the density operator in Eq. (17) for \( d = 0.1 \).

**B. Example: two qubits**

For two qubits, the joint probability of detecting a photon in the horizontal output ports of each PBS is given by
\[
    p_2(t) = \sum_{i_1,i_2=0}^{3} \frac{S_{i_1,i_2}}{4} \chi_{1,i_1} \chi_{2,i_2} \\
    = \frac{a_0^2}{2} + \sum_{f=1}^{3} (a_f \cos(\omega_f t) + b_f \sin(\omega_f t)),
\]
where in the second line, we have written the signal in terms of its Fourier coefficients. The extent of the summation depends on the specific choice of relative frequencies, and \( \omega_f \) are
functions of \( \omega_l \) and \( \omega_2 \) from the set \( \{ \omega_l, \omega_2, 2\omega_l, 2\omega_2, \omega_l \pm \omega_2, \omega_l \pm 2\omega_2, \omega_l \pm \omega_2 \} \). The elements of this set are not necessarily in order of size, and to relate them to \( \omega_l \), one needs to consider explicit values for \( \omega_l \) and \( \omega_2 \).

As an example, consider the two-qubit state,
\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( (|H\rangle|V\rangle + |R\rangle|L\rangle) \right). \tag{21}
\]

Wave-plate retardances of \( \beta = 11\pi/15 \) and a frequency ratio \( r = \omega_2/\omega_1 = 5 \) produces the signal shown in Fig. 4(b). The

\[
\rho_2 = \begin{pmatrix}
0.125 & 0.25 + 0.125i \\
0.25 - 0.125i & 0.625 \\
0.125i & -0.125 + 0.25i \\
0.125 & 0.25 + 0.125i
\end{pmatrix}, \tag{22}
\]

which corresponds to the density operator \( \hat{\rho}_2 = |\psi_2\rangle\langle\psi_2| \) where \( |\psi_2\rangle \) is defined in Eq. (21). In general, given separable qubits, \( p_2 \) factorizes into a product of \( p_1 \) for each qubit.

### III. SUMMARY & CONCLUDING REMARKS

We presented a scheme for performing quantum state tomography of photonic polarization-encoded multiqubit states. The scheme is simpler than traditional tomographic protocols in that only one wave plate and one polarizing beam splitter is required per photon mode.

In this scheme, photon-counting detectors measure a pseudocontinuous time-dependent joint probability as the wave plates rotate at frequency \( \Omega_m \). The Fourier coefficients of the signal give the Stokes’ parameters which describe the state. For a single qubit, the optimal wave-plate retardance is \( \beta \approx 11\pi/15 \).

This technique reduces the number of required optical elements and the experimental complexity scales linearly with the number of qubits, in terms of the number of settings required (wave-plate rotation frequencies) rather than exponentially, as is the case with QST that uses discrete measurement settings.

An open question is whether the representation of a quantum state as a continuous signal will provide intuitive means for establishing certain properties of the state such as its entanglement.

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### APPENDIX: PROBABILITY SIGNAL AND FOURIER COEFFICIENTS FOR TWO-QUBIT STATE

In this Appendix, we give the probability signal for the specific two-qubit example described in Sec. II B. We also provide expressions for the Fourier coefficients in terms of the Stokes’ parameters, as well as the inverted expressions for the Stokes’ parameters in terms of the Fourier coefficients.

The signal probability for the state,
\[
|\psi_2\rangle = \frac{1}{\sqrt{2}} \left( (|H\rangle|V\rangle + |R\rangle|L\rangle) \right), \tag{A1}
\]

with a frequency ratio \( r = \omega_2/\omega_1 = 5 \), is

\[
p_2(t) = \frac{a_0}{2} + b_1 \sin(\omega_1 t) + b_2 \sin(2\omega_1 t) + b_3 \sin(3\omega_1 t) + b_5 \sin(5\omega_1 t) + b_7 \sin(7\omega_1 t) + b_8 \sin(8\omega_1 t) + b_9 \sin(9\omega_1 t) + b_{10} \sin(10\omega_1 t) + b_{11} \sin(11\omega_1 t) + b_{12} \sin(12\omega_1 t) + a_2 \cos(2\omega_1 t) + a_3 \cos(3\omega_1 t) + a_4 \cos(4\omega_1 t) + a_6 \cos(6\omega_1 t) + a_7 \cos(7\omega_1 t) + a_8 \cos(8\omega_1 t) + a_{10} \cos(10\omega_1 t) + a_{11} \cos(11\omega_1 t) + a_{12} \cos(12\omega_1 t), \tag{A2}
\]

where the Fourier coefficients are

\[
a_0 = (c^2(S_{3,3} + S_{0,3} + S_{3,0}) + S_{0,0})/2, \tag{A3a}
\]
\[
a_2 = s^2(c^2S_{3,3} + S_{3,0})/4, \tag{A3b}
\]
\[
a_3 = -a_2 = cs^2S_{1,2}/4, \tag{A3c}
\]
\[
a_4 = -a_6 = c^2S_{2,2}/2, \tag{A3d}
\]
\[
a_8 = s^2(S_{1,1} + S_{3,3})/8, \tag{A3e}
\]
\[
a_{10} = -a_{11} = cs^3S_{2,1}/4, \tag{A3f}
\]
\[
a_{12} = s^4(S_{3,3} + S_{3,0})/4, \tag{A3g}
\]

and

\[
b_1 = cs(c^2S_{2,3} + S_{2,0})/2, \tag{A4a}
\]
\[
b_2 = s^2(c^2S_{1,3} + S_{1,0})/2, \tag{A4b}
\]
\[
b_3 = b_7 = cs^3S_{1,2}/4, \tag{A4c}
\]
\[
b_5 = cs(c^2S_{2,2} + S_{0,2})/2, \tag{A4d}
\]
\[
b_8 = s^4(S_{1,1} - S_{1,3})/8, \tag{A4e}
\]
\[ b_9 = -b_{11} = -cs^3 S_{2,3}/4, \quad (A4f) \]
\[ b_{10} = s^2(2s_3 + s_0)/4, \quad (A4g) \]
\[ b_{12} = s^4(2S_{1,3} + S_{3,1})/8, \quad (A4h) \]

where \( c = \cos(\beta/2) \) and \( s = \sin(\beta/2) \). Inverting Eqs. (A3) and (A4) gives

\[ S_{0,0} = 4c^2((a_8 + a_{12})c^2 - (a_2 + a_{10})s^2)^2/s^4 + 2a_0, \quad (A5a) \]
\[ S_{0,1} = -4(b_5 + b_{12})c^2 - b_{10}s^2)/s^4, \quad (A5b) \]
\[ S_{0,2} = 2b_5/c - 4b_3c/s^3, \quad (A5c) \]
\[ S_{0,3} = 4a_{10}/s^2 - 4(a_8 + a_{12})c^2/s^4, \quad (A5d) \]
\[ S_{1,0} = 4((b_8 - b_{12})c^2 + b_3s^2)/s^4, \quad (A5e) \]
\[ S_{1,1} = 4(a_8 - a_{12})/s^4, \quad (A5f) \]
\[ S_{1,2} = 4a_3/c^3, \quad (A5g) \]

Substituting the Stokes’ parameters, along with the Fourier coefficients in Fig. 4(b), into Eq. (2), gives the reconstructed density matrix in Eq. (22) which corresponds to the density operator \( \hat{\rho}_2 = |\psi_2\rangle \langle \psi_2| \) where \( |\psi_2\rangle \) is defined in Eq. (A1).